SEMESTRAL

Elementary Number Theory

Instructor: Ramdin Mawia   Marks: **50**   **Time:** November 13, 2024; 10:00–13:00.

Attempt FIVE problems. The maximum you can score is 50.

1. Prove or disprove: **7+8=15**

   i. For all positive integers n, the congruence $(x^2-7)(x^2-13)(x^2-91) \equiv 0 \pmod{n}$ is solvable.

   ii. Given any integer $k \geqslant 1$, there exists an integer $n \geqslant 1$ such that $\mu(n) = \mu(n+1) = \cdots = \mu(n+k)$. [*Hint.* Can this common value be ± 1?]

2. State the Reciprocity Laws for the Legendre and Jacobi symbols. Use them to compute $(1997/10005)$. **10**

3. Does the congruence $7X^2 + X + 3 \equiv 0 \pmod{317}$ have a solution? If yes, solve it. **10**

4. Describe all the primes in the set $S := \{x^2 + 3y^2 : x, y \in \mathbb{Z}\}$. Give three examples of primes in S with the corresponding values of x and y. [*Hint.* Think of class numbers!] **10**

OR

4′. Let $f(X, Y)$ be a primitive binary quadratic form, and let $k > 1$ be a squarefree integer. Prove that for any divisor $d > 0$ of k, there are positive integers x and y such that $d = \gcd(x, y) = \gcd(f(x, y), k)$.

5. Let Λ be the von Mangoldt function. Describe the set $Z = \{n \geqslant 1 : (\Lambda * \Lambda)(n) = 0\}$. **10**

OR

5′. Prove that

$$\sum_{n \leqslant x} \mu(n) \left[\frac{x}{n}\right] = 1$$

for all $x \geqslant 1$. [*Hint.* Write $[y] = \sum_{m \leqslant y} 1$ and interchange the order of summation. Dirichlet's hyperbola method is also applicable!]

6. Prove that **10**

$$\sum_{p \leqslant x} \frac{1}{p} > \log\log x - 1/2$$

for $x \geqslant 2$, where the sum runs through the primes $p \leqslant x$.

OR

6′. Let p_n denote the nth prime. Show that $p_{n+1} \leqslant p_1 p_2 \cdots p_n + 1$ for all $n \geqslant 1$. Deduce that $p_n < 2^{2^n}$ for all $n \geqslant 1$ and that $\pi(x) \geqslant \log\log x$ for all $x \geqslant 2$.

7. Let $(u_n)_{n \geqslant 0}$ be a complex sequence satisfying a linear recurrence of order 2. Suppose $u_0 = 0, u_1 = 2, u_2 = 16, u_3 = 98$. Find the nth term u_n of the sequence in closed form. **10**

The End